Higher order Hardy inequalities

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Abstract

This note deals with the inequality

\[
\left( \int_a^b |u(x)|^q w_0(x) \, dx \right)^{1/q} \leq C \left( \int_a^b |u^{(k)}(x)|^p w_k(x) \, dx \right)^{1/p},
\]

more precisely, with conditions on the parameters \(p > 1, q > 0\) and on the weight functions \(w_0, w_k\) (measurable and positive almost everywhere) which ensure that (1) holds for all functions \(u\) from a certain class \(K\) with a constant \(C > 0\) independent of \(u\).

Here \(-\infty \leq a < b \leq \infty\) and \(k \in \mathbb{N}\) and we will consider classes \(K\) of functions \(u = u(x)\) defined on \((a, b)\) whose derivatives of order \(k - 1\) are absolutely continuous and which satisfy the “boundary conditions”

\[
\begin{align*}
  u^{(i)}(a) &= 0 \quad \text{for} \quad i \in M_0, \\
  u^{(j)}(b) &= 0 \quad \text{for} \quad j \in M_1,
\end{align*}
\]

where \(M_0, M_1\) are subsets of the set \(M = \{0, 1, \ldots, k - 1\}\); we will suppose that the number of conditions in (2) is exactly \(k\). This class will be denoted by

\[\text{AC}^{(k-1)}(a, b; M_0, M_1).\]

The conditions (2) are reasonable since they allow to exclude functions like polynomials of order \(\leq k - 1\) for which the right hand side in (1) is zero while the left hand side is positive.
Let us start with some remarks.

(i) We will concentrate on the case

\[ k > 1 \]  \hfill (4)

since for \( k = 1 \) the problem is completely solved: see, e.g., the book Opic, Kufner [4], Chapter 1. Some particular results concerning the case \( k = 2, k = 3 \) and - for a special choice of the sets \( M_0, M_1 \) - also higher values of \( k \) can be found in the paper Kufner, Wannebo [3].

(ii) For \((a, b) = (0, \infty), k \in \mathbb{N}\) arbitrary and \( M_0 = M, M_1 = \emptyset \) or \( M_0 = \{0, 1, \ldots, m - 1\}, M_1 = M \setminus M_0, 0 < m < k \), the problem is also solved: see Stepanov [5] or Kufner, Heinig [2], respectively. These results cover all reasonable cases when the interval \((a, b)\) is infinite. Therefore, we will concentrate on the case of a finite interval \((a, b)\). Without loss of generality it can be assumed that

\[ (a, b) = (0, 1). \]  \hfill (5)

In the sequel, we will make substantial use of some functions and constants. For \( r \neq 1 \), we will denote

\[ r' = \frac{r}{r - 1}, \text{ i.e. } \frac{1}{r} + \frac{1}{r'} = 1. \]

Further, let us denote for \( i = 1, 2 \)

\[ W_{0i}(t) = w_0(t)^{t^{\alpha_i} q} (1 - t)^{\beta_i q}, \]

\[ W_{ki}(t) = w_k^{1-p'}(t)^{t^{\gamma_i} p'} (1 - t)^{\delta_i p'}, \]  \hfill (6)

where \( w_0(t), w_k(t) \) are the weight functions appearing in (1) and \( \alpha_i, \beta_i, \gamma_i, \delta_i \) \((i = 1, 2)\) are certain nonnegative integers, and let us introduce functions

\[ B_1(x) = \left( \int_x^1 W_{01}(t) dt \right)^{1/q} \left( \int_0^x W_{k1}(t) dt \right)^{1/p'}, \]  \hfill (7)

\[ B_2(x) = \left( \int_0^x W_{02}(t) dt \right)^{1/q} \left( \int_x^1 W_{k2}(t) dt \right)^{1/p'}, \]  \hfill (8)

and constants

\[ A_1 = \left[ \int_0^1 \left( \int_x^1 W_{01}(t) dt \right)^{r'/q} \left( \int_0^x W_{k1}(t) dt \right)^{r'/q'} W_{k1}(x) dx \right]^{1/r}, \]  \hfill (9)

\[ A_2 = \left[ \int_0^1 \left( \int_0^x W_{02}(t) dt \right)^{r'/q} \left( \int_x^1 W_{k2}(t) dt \right)^{r'/q'} W_{k2}(x) dx \right]^{1/r}. \]  \hfill (10)
where
\[ \frac{1}{r} = \frac{1}{q} - \frac{1}{p}. \] (11)

We suppose that all expressions appearing in formulas (7) - (10) are well defined. Of course, it also depends on the values \( \alpha_i, \beta_i, \gamma_i, \delta_i \) which have not yet been determined. Later, we will show how these integers can be determined by the sets \( M_0 \) and \( M_1 \) which appear in the conditions (2).

If we suppose for a moment that these integers are known, then the main result can be formulated as follows:

**Proposition 1**

Let \( M_0, M_1 \) be two nonempty subsets of the set \( \{0, 1, \ldots, k-1\} \) containing together \( k \) elements. Let \( \alpha_i, \beta_i, \gamma_i, \delta_i, \ i = 1, 2, \) be nonnegative integers corresponding to the pair \( M_0, M_1 \). Let \( w_0(t), w_k(t) \) be weight functions defined on \((0, 1)\) and let

\[ 1 < p < \infty, \ 0 < q < \infty, \ q \neq 1. \]

Then the (HARDY) inequality

\[ \left( \int_0^1 |u(t)|^q w_0(t) dt \right)^{1/q} \leq C \left( \int_0^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{1/p} \] (12)

holds for every function \( u \in AC^{(k-1)}(0, 1) \) satisfying the conditions

\[ u^{(i)}(0) = 0 \quad \text{for} \quad i \in M_0, \] (13)

\[ u^{(j)}(1) = 0 \quad \text{for} \quad j \in M_1 \]

if and only if

\[ \sup_{0<x<1} B_i(x) = B_i < \infty, \quad i = 1, 2 \] (14)

in the case \( p \leq q \), and

\[ A_i < \infty, \quad i = 1, 2 \] (15)

in the case \( p > q \), where \( B_i(x) \) and \( A_i \) are given by formulas (7) - (11).
Then we have

\[ u^{(k)} = f \quad \text{in} \quad (0, 1), \]
\[ u^{(i)}(0) = 0 \quad \text{for} \quad i \in M_0, \]
\[ u^{(j)}(1) = 0 \quad \text{for} \quad j \in M_1 \]

where \( f \) does not change the sign in \((0, 1)\) and \(M_0, M_1\) are the subsets of \(M = \{0, 1, \ldots, k - 1\}\) mentioned in Proposition 1.

Suppose that the solution \( u \) can be expressed uniquely in the form

\[ u(x) = \int_0^x K_1(x,t)f(t)dt + \int_x^1 K_2(x,t)f(t)dt. \]

The kernels \(K_1(x,t), K_2(x,t)\) are then polynomials. We will write

\[ K_i(x,t) \approx x^{\alpha_i}(1-x)^{\beta_i}t^{\gamma_i}(1-t)^{\delta_i} \]

if there exist positive constants \(c_1, c_2\) such that the estimates

\[ c_1 \leq \frac{K_i(x,t)}{x^{\alpha_i}(1-x)^{\beta_i}t^{\gamma_i}(1-t)^{\delta_i}} \leq c_2 \]

hold for \(0 < t < x < 1\) \((i = 1)\) and \(0 < x < t < 1\) \((i = 2)\), respectively.

Now, we will show under what conditions (18) is fulfilled. For this purpose, let us split the set \(M = \{0, 1, \ldots, k - 1\}\) into \(s\) successive groups \(G_1, G_2, \ldots, G_s\) \((s \geq 2)\) according to the following scheme:

\[ G_1 = \{0, 1, \ldots, m - 1\} \quad (k_1 \text{ elements, } k_1 = m), \]
\[ G_2 = \{m, m + 1, \ldots, n - 1\} \quad (k_2 \text{ elements, } k_2 = n - m), \]
\[ G_3 = \{n, n + 1, \ldots, r - 1\} \quad (k_3 \text{ elements, } k_3 = r - n), \]
\[ \ldots \]
\[ G_s = \{h, h + 1, \ldots, k - 1\} \quad (k_s \text{ elements, } k_s = k - h), \]

(i.e. \(G_i\) has \(k_i\) elements, \(k_i > 0\), \(i = 1, 2, \ldots, s\), and \(k_1 + k_2 + \ldots + k_s = k\)), and suppose that the sets \(M_0\) and \(M_1\) appearing in the boundary conditions in (16) are defined as follows:

\[ M_0 = G_1 \cup G_2 \cup \ldots \cup G_{s-1}, \quad M_1 = G_2 \cup G_4 \cup \ldots \cup G_s \quad \text{for } s \text{ even.} \]
\[ M_0 = G_1 \cup G_2 \cup \ldots \cup G_s, \quad M_1 = G_2 \cup G_4 \cup \ldots \cup G_{s-1} \quad \text{for } s \text{ odd.} \]

Then we have
Proposition 2

If the set \( M = \{0, 1, \ldots, k-1\} \) is split into \( s \) groups according to (19), the sets \( M_0 \) and \( M_1 \) are defined by (20) and (21) and the solution \( u \) to the boundary value problem (16) can be expressed in the form (17), then

\[
K_1(x, t) \approx x^{k_1-1}t^{k_2} \quad K_2(x, t) \approx x^{k_1}t^{k_2-1} \quad \text{for } s = 2, \\
K_i(x, t) \approx x^{k_1}(1-t)^{k_i} \quad i = 1, 2, \text{ for } s \text{ odd}, \\
K_i(x, t) \approx x^{k_1}t^{k_i} \quad i = 1, 2, \text{ for } s > 2 \text{ even}.
\]

Remarks. (i) The proof of Proposition 2 is elementary but cumbersome. It is based on the fact that the solution \( u \) to the boundary value problem (16) can be expressed in the form

\[
u(x) = c_0 \int_0^x (x - t_1)^{k_1-1} \int_{t_1}^1 (t_2 - t_1)^{k_2-1} \int_{t_2}^{t_3} (t_3 - t_2)^{k_3-1} \cdots \\
\cdots F(t_{s-1}) dt_{s-1} \cdots dt_2 dt_1
\]

where \( c_0 = [(k_1-1)!(k_2-1)! \cdots (k_s-1)!]^{-1} \) and \( F(t_{s-1}) \) is either

\[
\int_{t_{s-1}}^1 (t_s - t_{s-1})^{k_s-1} f(t_s) dt_s \quad \text{for } s \text{ even}
\]

or

\[
\int_0^{t_{s-1}} (t_{s-1} - t_s)^{k_s-1} f(t_s) dt_s \quad \text{for } s \text{ odd}.
\]

For \( s = 2 \), it can be found in the paper [3], for \( s > 2 \) in the preprint [1].

(ii) In (20), (21) we have always assumed that the first group \( G_1 \) belongs to \( M_0 \) so that we start with the boundary condition \( u(0) = 0, 0 \in M_0 \). If we suppose that \( 0 \in M_1 \), i.e. that the boundary condition \( u(1) = 0 \) appears in (16), and have

\[
M_0 = G_2 \cup G_4 \cup \ldots, \quad M_1 = G_1 \cup G_3 \cup \ldots,
\]

then we simply exchange the role of the sets \( M_0 \) and \( M_1 \), i.e. of the endpoints \( x = 0 \) and \( x = 1 \), and a corresponding assertion holds again, if we replace in (22) \( x \) by \( 1 - x \) and \( t \) by \( 1 - t \).

(iii) In the foregoing cases, we have assumed that

\[
M_0 \cup M_1 = M, \quad \text{i.e. } M_0 \cap M_1 = \emptyset.
\]
If the sets $M_0$ and $M_1$ again have together $k$ elements, but have a nonempty intersection, then the method described above cannot be used. Nonetheless, many examples allow to expect that - provided there is a unique representation of the solution $u$ of (16) in the form (17) - the kernels $K_i(x,t)$ again behave according to (18). Therefore, let us formulate the following conjecture:

Suppose that $M_0 \cap M_1 \neq \emptyset$.

(a) Define $\tilde{M}_1$ by

$$\tilde{M}_1 = M \setminus M_0.$$ 

Then the pair $M_0, \tilde{M}_1$ satisfies the conditions of either Proposition 2 (if $G_1 \subset M_0$) or of part (ii) of this Remark (if $G_1 \subset \tilde{M}_1$), and consequently, the kernels $K_i^{(a)}(x,t)$ corresponding to the pair $M_0, \tilde{M}_1$ satisfy (18): There are positive integers $\alpha_i(a), \beta_i(a), \gamma_i(a), \delta_i(a)$ such that

$$K_i^{(a)}(x,t) \approx x^{\alpha_i(a)}(1 - x)^{\beta_i(a)} t^{\gamma_i(a)}(1 - t)^{\delta_i(a)}, \quad i = 1, 2.$$ 

(b) Define $\tilde{M}_0$ by

$$\tilde{M}_0 = M \setminus M_1.$$ 

Then the pair $\tilde{M}_0, M_1$ again satisfies the conditions which allow to state that for the corresponding kernels $K_i^{(b)}(x,t)$ we have

$$K_i^{(b)}(x,t) \approx x^{\alpha_i(b)}(1 - x)^{\beta_i(b)} t^{\gamma_i(b)}(1 - t)^{\delta_i(b)}, \quad i = 1, 2.$$ 

(c) For the kernels $K_i(x,t)$ corresponding to the initial pair $M_0, M_1$ we have (18) with

$$\alpha_i = \alpha_i(a), \beta_i = \beta_i(b), \gamma_i = \gamma_i(a), \delta_i = \delta_i(b).$$

Idea of the proof of Proposition 1

We consider the Hardy inequality (12) on the class $AC^{(k-1)}(0,1; M_0, M_1)$, i.e., for functions $u$ satisfying (13). Therefore, let us consider the boundary value problem (16) and denote by $T$ the operator defined by formula (17):

$$(Tf)(x) = \int_0^x K_1(x,t)f(t)dt + \int_x^1 K_2(x,t)f(t)dt.$$ 

Since the function $u = Tf$ satisfies conditions (13) and $u^{(k)} = f$, we can instead of the inequality (12) investigate the inequality

$$\left( \int_0^1 |(Tf)(x)|^q w_0(x)dx \right)^{1/q} \leq C \left( \int_0^1 f^p(x)w_k(x)dx \right)^{1/p}$$ 

for functions $f \geq 0$. 

\[ (23) \]
Now, it can be shown that the validity of (23) for $f \geq 0$ is equivalent to the validity of the inequalities
\[
\left(\int_0^1 |(J_i f)(x)|^q w_0(x)dx\right)^{1/q} \leq C_i \left(\int_0^1 f^p(x)w_k(x)dx\right)^{1/p}, \quad i = 1, 2, \tag{24}
\]
where
\[
(J_1 f)(x) = \int_0^x K_1(x, t)f(t)dt, \quad (J_2 f)(x) = \int_x^1 K_2(x, t)f(t)dt.
\]
But due to (18), the inequalities (24) are equivalent to the inequalities
\[
\left(\int_0^1 \left(x^{\alpha_1}(1-x)^{\beta_1} \int_0^x t^{\gamma_1}(1-t)^{\delta_1} f(t)dt\right)^q w_0(x)dx\right)^{1/q} \leq \tilde{C}_1 \left(\int_0^1 f^p(x)w_k(x)dx\right)^{1/p}
\]
and
\[
\left(\int_0^1 \left(x^{\alpha_2}(1-x)^{\beta_2} \int_x^1 t^{\gamma_2}(1-t)^{\delta_2} f(t)dt\right)^q w_0(x)dx\right)^{1/q} \leq \tilde{C}_2 \left(\int_0^1 f^p(x)w_k(x)dx\right)^{1/p}
\]
respectively, and these last two inequalities can be easily rewritten into the form
\[
\left(\int_0^1 |(Hg)(x)|^q w(x)dx\right)^{1/q} \leq \tilde{C} \left(\int_0^1 g^p(x)v(x)dx\right)^{1/p}, \tag{25}
\]
where $H$ is the Hardy operator,
\[
(Hg)(x) = \int_0^x g(t)dt \quad \text{or} \quad (Hg)(x) = \int_x^1 g(t)dt.
\]
Finally, necessary and sufficient conditions for the validity of (25) (see, e.g., [4]) lead to the conditions (14) (if $p \leq q$) or (15) (if $p > q$).

Consequently, the integers $\alpha_1, \ldots, \delta_2$ which appear in (6) can be determined from the behavior of the kernels $K_1(x, t), K_2(x, t)$ described by (18). $\Box$
References