Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. (2006), 127–139
Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations
El Escorial, Madrid (Spain), June 21-25, 2004

A boundary integral equation for Calderón's inverse conductivity problem

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Abstract

Towards a constructive method to determine an L^∞-conductivity from the corresponding Dirichlet to Neumann operator, we establish a Fredholm integral equation of the second kind at the boundary of a two dimensional body. We show that this equation depends directly on the measured data and has always a unique solution. This way the geometric optics solutions for the L^∞-conductivity problem can be determined in a stable manner at the boundary and outside of the body.

1. Introduction

Calderón's inverse conductivity problem is to determine the coefficients of an elliptic differential equation from the corresponding boundary data, i.e. from the Dirichlet to Neumann operator. More precisely, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with connected complement and $\sigma : \Omega \to (0, \infty)$ is measurable and bounded away from zero and infinity. Let $u \in H^1(\Omega)$ be the unique solution to

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \tag{1.1}$$

$$u\big|_{\partial\Omega} = \phi \in H^{1/2}(\partial\Omega). \tag{1.2}$$

Keywords: Inverse conductivity problem, quasiregular mappings, complex geometric optics solutions.

MSC2000: 35R30.

* The research of both authors is supported by the Academy of Finland.

The inverse conductivity problem is to recover σ from the map

$$\Lambda_\sigma : \phi \mapsto \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

Here ν is the unit outer normal to the boundary and the derivative $\sigma\partial u/\partial\nu$ exists as an element of $H^{-1/2}(\partial\Omega)$, defined by

$$\left\langle \sigma \frac{\partial u}{\partial \nu}, \psi \right\rangle = \int_\Omega \sigma \nabla u \cdot \nabla \psi \, dm, \tag{1.3}$$

where $\psi \in H^1(\Omega)$ and dm denotes the Lebesgue measure.

In a recent work we presented a solution to this problem in two dimensions, for details see [2]. The proof made strong use of the so called geometric optics solutions to (1.1). These have the special asymptotics

$$u(z) = u_k(z) = e^{ikz}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \text{ as } |z| \to \infty \tag{1.4}$$

at infinity. Here k is a complex parameter. Since these solutions are globally defined we need to extend σ to the whole plane by setting $\sigma(z) \equiv 1$ for $z \notin \Omega$.

The starting point towards Calderón problem is to show that Λ_σ determines firstly the values of u_k on $\partial\Omega$ and hence in the complement $\mathbb{C} \setminus \Omega$ and secondly, it determines a scattering coefficient or the so called non- linear Fourier transform $\tau_\mu(k)$. Using these quantities it was then shown that Λ_σ determines the solutions u_k also inside Ω, and this readily determines the coefficient σ, see [2].

In the present paper we continue this work and give a constructive method for finding the boundary values of the geometric optics solutions u_k, directly from the Dirichlet to Neumann operator. For this purpose it is natural to seek for an integral equation on the boundary of the domain, depending on σ only through Λ_σ, that will yield the boundary values of the geometric optics solutions. In the smooth case this was done in [4] for dimensions three and higher and in [3] for the dimension two. Here we establish for L^∞-conductivities in the plane a natural but somewhat different integral equation than in [3]. This approach will also describe a constructive method for determining the transform $\tau_\mu(k)$ directly from Λ_σ.

For simplicity, we assume that $\sigma \equiv 1$ near the boundary of Ω and we give the proof in the case $\Omega = \mathbb{D}$, the unit disk $\mathbb{D} = \{z : |z| < 1\}$. To explain the results in detail recall first from [2] that it is convenient to replace (1.1) by an equivalent first order system. In the complex notation we may write $f = u + iv$, so that the system obtains the form

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{1.5}$$

where $\mu = (1-\sigma)/(1+\sigma)$. In particular, note that μ is real valued and that (1.5) is only $\mathbb{R}$-linear. The assumptions for σ imply that $\|\mu\|_{L^\infty} \leq \kappa < 1$. Furthermore, the "conjugate" function v is obtained from the identity

$$\partial_T v\,(z) = \Lambda_\sigma u\,(z), \qquad z \in \partial\mathbb{D}.$$

In brief, we are now looking for a solution $f = f_\mu$ to (1.5) with the asymptotics

$$f_\mu(z,k) = e^{ikz}\,(1 + \mathcal{O}(1/z)) \text{ as } |z| \to \infty. \tag{1.6}$$

This approach leads naturally to the concept of the μ-Hilbert transform $\mathcal{H}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ defined by

$$\mathcal{H}_\mu u = v.$$

That is, the tangential derivative gives

$$\partial_T \mathcal{H}_\mu u = \Lambda_\sigma u \tag{1.7}$$

and thus basically the μ-Hilbert transform is just a reformulation of the Dirichlet-to-Neumann operator Λ_σ.

In case $\sigma \equiv 1$, or $\mu \equiv 0$, we have $\mathcal{H}_\mu = \mathcal{H}_0$, the usual Hilbert transform on the unit circle. This is a singular integral operator with Fourier multiplier $m(\xi) = -i\xi/|\xi|$ for $\xi \in \mathbb{Z} \setminus \{0\}$ and $m(0) = 0$, so that

$$\widehat{\mathcal{H}_0 g}(\xi) = m(\xi)\widehat{g}(\xi), \qquad g \in L^2(\partial\mathbb{D}).$$

The Hilbert transform $\mathcal{H}_0$ determines the Riesz projections onto the Hardy spaces on $\partial\mathbb{D}$,

$$\mathcal{P}_0 g = \frac{1}{2}(I + i\mathcal{H}_0)g + \frac{1}{2}\fint_{\partial\mathbb{D}} g\,ds,$$

where

$$\fint_{\partial\mathbb{D}} g\,ds = \frac{1}{2\pi}\int_{\partial\mathbb{D}} g\,ds.$$

Similarly the new Hilbert transforms determine the projections

$$\mathcal{P}_\mu g = \frac{1}{2}(I + i\mathcal{H}_\mu)g + \frac{1}{2}\fint_{\partial\mathbb{D}} g\,ds, \tag{1.8}$$

and in fact, the range of $\mathcal{P}_\mu$ consists of functions analytic with respect to a new complex structure determined by σ. For an explicit description see Section 2. Note also that we use slightly different notations than in [2].

Finally we need to conjugate the projections with the exponential functions and set

$$\mathcal{P}_\mu^k(g)(z) = e^{ikz}\mathcal{P}_\mu(e^{-ik\,\cdot}g)(z), \qquad g \in H^{1/2}(\partial\mathbb{D}). \tag{1.9}$$

With these notations the main theorem of this paper reads as

Theorem 1.1

Assume that $\operatorname{supp}(\mu) \subset \mathbb{D}$. *Then*

a) For each $k \in \mathbb{C}$ *the operator* $I - \left(\mathcal{P}_\mu + \mathcal{P}_0^k\right)$ *is invertible on* $H^{1/2}(\partial\mathbb{D})$.

b) The function $f = f_\mu$ *from the equations* (1.5) *and* (1.6) *satisfies on the boundary the singular integral equation*

$$f(z) + e^{ikz} = (\mathcal{P}_\mu + \mathcal{P}_0^k)f\,(z), \qquad z \in \partial\mathbb{D}. \tag{1.10}$$

In fact, we prove that the operator $I - \left(\mathcal{P}_\mu + \mathcal{P}_0^k\right)$ is a Fredholm operator with index zero. By (1.7) and (1.8) this Fredholm operator depends explicitly on the Dirichlet-to-Neumann operator Λ_σ.

The methods we will use are complex analytic, but it turns out that the outcome, and the operator $\mathcal{P}_\mu$ in particular, is $\mathbb{R}$- linear. Moreover, in (1.10) the dependence on μ or σ arises only through this operator. It is natural to write this $\mathbb{R}$-linear operator in the vector formulation, under the identification $\mathbb{R}^2 = \mathbb{C}$, and then the μ-dependence is even more explicit,

$$\mathcal{P}_\mu(f) = \frac{1}{2}\begin{pmatrix} I + L & (\mathcal{H}_\mu + L)^{-1} - L \\ \mathcal{H}_\mu & I + L \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad f = u + iv \tag{1.11}$$

where

$$L(g) = \fint_{\partial\mathbb{D}} g\, ds$$

is the average operator.

The solution $f_\mu \in H^1_{\mathrm{loc}}(\mathbb{C})$ to (1.5) and (1.6) exists and is unique by [2, Theorem 4.2]. Furthermore, f_μ is locally Hölder continuous and analytic outside the disk $\mathbb{D}$. Writing $f_\mu = e^{ikz}m_\mu$, the factor m_μ has the development

$$m_\mu(z) = m_\mu(z,k) = 1 + \frac{a_1(k)}{z} + \frac{a_2(k)}{z^2} + \cdots \text{ as } |z| \to \infty. \tag{1.12}$$

Thus it follows that the Fredholm equation (1.10) determines the solutions $f_\mu(z,k)$ for $|z| \geq 1$ and $k \in \mathbb{C}$. Finally, this information determines also τ_μ, the non-linear Fourier transform of μ; see Section 4.

2. Projections and Hilbert transforms

We will use the complex analytic approach with the corresponding notations. In particular, we let $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$; when clarity requires we write $\overline{\partial} = \partial_{\overline{z}}$ or $\partial = \partial_z$.

Let us begin by explaining the equivalence of the equations (1.1) and (1.5). If $u \in H^1(\Omega)$ is a real solution of (1.1) then there exists a real function $v \in H^1(\Omega)$, called the *σ-harmonic conjugate* of u, such that $f = u + iv$ satisfies the $\mathbb{R}$- linear Beltrami equation

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{2.1}$$

where $\mu = (1-\sigma)/(1+\sigma)$. Indeed, we have the following simple lemma.

Lemma 2.1

Assume $u \in H^1(\mathbb{D})$ is real valued and satisfies the conductivity equation (1.1). Then there exists a function $v \in H^1(\mathbb{D})$, unique up to a constant, such that $f = u + iv$ satisfies the $\mathbb{R}$-linear Beltrami equation

$$\overline{\partial} f = \mu \overline{\partial f}, \tag{2.2}$$

where $\mu = (1-\sigma)/(1+\sigma)$.

Conversely, if $f \in H^1(\mathbb{D})$ satisfies (2.2) with a $\mathbb{R}$-valued μ, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy

$$\nabla \cdot \sigma \nabla u = 0 \quad \textit{and} \quad \nabla \cdot \frac{1}{\sigma} \nabla v = 0, \tag{2.3}$$

respectively, where $\sigma = (1-\mu)/(1+\mu)$.

Proof. Denote by w the vectorfield

$$w = (-\sigma \partial_2 u, \sigma \partial_1 u) \in L^2(\mathbb{D})$$

where $\partial_1 = \partial/\partial x$ and $\partial_2 = \partial/\partial y$ for $z = x + iy \in \mathbb{C}$. Then by (1.1) the integrability condition $\partial_2 w_1 = \partial_1 w_2$ holds for the distributional derivatives. Therefore there exists $v \in H^1(\mathbb{D})$, unique up to a constant, such that

$$\partial_1 v = -\sigma \partial_2 u \tag{2.4}$$

$$\partial_2 v = \sigma \partial_1 u. \tag{2.5}$$

It is a simple calculation to see that this is equivalent to (2.2). □

Since the function v in Lemma 2.1 is defined only up to a constant we will normalize it by assuming

$$L(v) = \int_{\partial \mathbb{D}} v \, ds = 0. \tag{2.6}$$

This way we obtain a unique map $\mathcal{H}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ by setting

$$\mathcal{H}_\mu : u\big|_{\partial\mathbb{D}} \mapsto v\big|_{\partial\mathbb{D}}. \tag{2.7}$$

The function v satisfying (2.4), (2.5) and (2.6) is called the *σ-harmonic conjugate* of u and $\mathcal{H}_\mu$ the *Hilbert transform* corresponding to equation (2.2) or, in brief, the μ-Hilbert transform.

To connect the Hilbert transforms to the Dirichlet to Neumann operators, we show that

$$\partial_T \mathcal{H}_\mu(u) = \Lambda_\sigma(u). \tag{2.8}$$

Indeed, choose the counter clock-wise orientation for $\partial\mathbb{D}$ and denote by ∂_T the tangential (distributional) derivative on $\partial\mathbb{D}$ corresponding to this orientation. By the definition of Λ_σ we have

$$\int_{\partial\mathbb{D}} \varphi \Lambda_\sigma u \, ds = \int_{\mathbb{D}} \nabla\varphi \cdot \sigma \nabla u \, dm, \quad \varphi \in C^\infty(\overline{\mathbb{D}}).$$

Thus, by (2.4), (2.5) and integration by parts, we get

$$\int_{\partial\mathbb{D}} \varphi \Lambda_\sigma u = \int_{\mathbb{D}} (\partial_1\varphi\partial_2 v - \partial_2\varphi\partial_1 v)\ dm$$
$$= -\int_{\partial\mathbb{D}} v \partial_T \varphi \, ds$$

and we see that (2.8) holds in the weak sense.

So far we have only defined $\mathcal{H}_\mu(u)$ for real-valued u. Note that here $v = \mathcal{H}_\mu(u)$ is the real part of the function $g = -if$ satisfying $\overline{\partial} g = -\mu\overline{\partial g}$. Therefore it is natural to set

$$\mathcal{H}_\mu(iv) = i\mathcal{H}_{-\mu}(v) \tag{2.9}$$

and thus extend the definition of $\mathcal{H}_\mu(g)$ to all $\mathbb{C}$-valued $g \in H^{1/2}(\partial\mathbb{D})$. However, $\mathcal{H}_\mu(g)$ remains only $\mathbb{R}$- linear. Moreover, we have

$$\mathcal{H}_\mu \circ \mathcal{H}_{-\mu}\, u = \mathcal{H}_{-\mu} \circ \mathcal{H}_\mu\, u = -u + \fint_{\partial\mathbb{D}} u \, ds. \tag{2.10}$$

Lastly, in analogy with the Riesz projections we define the operator $\mathcal{P}_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$,

$$\mathcal{P}_\mu\, g = \frac{1}{2}(I + i\mathcal{H}_\mu)g + \frac{1}{2}\fint_{\partial\mathbb{D}} g \, ds. \tag{2.11}$$

Since with (2.10) we have $\mathcal{H}_{-\mu} - L = -(\mathcal{H}_\mu + L)^{-1}$ the formula (1.11) follows by taking the real and imaginary parts of the identity (2.11).

The subject of this paper is to understand the projection operators $\mathcal{P}_\mu$ and in particular to show that even if μ and $\tilde{\mu}$ were far apart the projections $\mathcal{P}_\mu$ and $\mathcal{P}_{\tilde{\mu}}$ are always close to each other. For instance their difference is shown to be a compact operator, and this fact will play a central role in the proof of the Fredholm properties of Theorem 1.1.

We begin with the simple

Lemma 2.2

The operator $\mathcal{P}_\mu$ is a projection, that is $\mathcal{P}_\mu^2 = \mathcal{P}_\mu$. Moreover, if $g \in H^{1/2}(\partial\mathbb{D})$, the following conditions are equivalent,

a) $g = f\big|_{\partial\mathbb{D}}$, where $f \in H^1(\mathbb{D})$ and satisfies (2.2);

b) $\mathcal{P}_\mu(g) = g$.

Proof. That $\mathcal{P}_\mu^2 = \mathcal{P}_\mu$ follows immediately from (2.9) and (2.10). Let then $g = u + iv \in H^{1/2}(\partial\mathbb{D})$. Taking separately the real and imaginary parts from (2.11) we see that $\mathcal{P}_\mu(g) = g$ if and only if

$$v = \mathcal{H}_\mu u + \frac{1}{2\pi}\int_{\partial\mathbb{D}} v\, ds$$

and

$$u = -\mathcal{H}_{-\mu} v + \frac{1}{2\pi}\int_{\partial\mathbb{D}} u\, ds;$$

in view of (2.10) these last equations are actually equivalent.

By definition of the Hilbert transforms, $v = \mathcal{H}_\mu u + C_0$ if and only if $u + iv$ extends to $\mathbb{D}$ as a solution to (2.2). □

With the above lemma we see that the range of $\mathcal{P}_\mu$ consists of the boundary values of the solutions to the equation $\overline{\partial} f = \mu\overline{\partial f}$. This is in complete analogy with the case $\mu \equiv 0$ where $\mathcal{P}_0$ is the projection to the boundary values of functions analytic in the disk, that is boundary values of the solutions to the equation $\overline{\partial} f = 0$.

The subspace complementary to range $(\mathcal{P}_\mu)$ admits an equally simple projection,

$$\mathcal{Q}_\mu g = \frac{1}{2}(I - i\mathcal{H}_\mu)g - \frac{1}{2}⨍_{\partial\mathbb{D}} g\, ds. \tag{2.12}$$

Since clearly

$$\mathcal{P}_\mu + \mathcal{Q}_\mu = I \tag{2.13}$$

the above identity (2.12) does define a projection with range equal to $\ker(\mathcal{P}_\mu)$. This could, of course, be verified also by a direct calculation as in Lemma 2.2.

By the definitions of the Hilbert transforms and the average operator we have $L\mathcal{H}_\mu = \mathcal{H}_\mu L = 0$ on $H^{1/2}(\partial\mathbb{D})$. Thus the formulae (2.11) and (2.12) give

$$\mathcal{P}_\mu L = L = L\mathcal{P}_\mu \text{ and } \mathcal{Q}_\mu L = 0 = L\mathcal{Q}_\mu. \tag{2.14}$$

In addition, as for the Riesz projections $\mathcal{P}_0$, $\mathcal{Q}_0$ the projection operators $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ are complex conjugate to each other, up to a constant term.

Lemma 2.3

For each $g \in H^{1/2}(\partial\mathbb{D})$ we have

$$\overline{\mathcal{P}_\mu(\overline{g})} = \mathcal{Q}_\mu(g) + L(g).$$

Proof. The property (2.9) shows that

$$\overline{\mathcal{H}_\mu(u - iv)} = \mathcal{H}_\mu(u) + i\mathcal{H}_{-\mu}(v) = \mathcal{H}_\mu(u + iv).$$

Therefore

$$\overline{\mathcal{P}_\mu(\overline{g})} = \frac{1}{2}g - \frac{i}{2}\,\overline{\mathcal{H}_\mu(\overline{g})} + \frac{1}{2}L(g) = \mathcal{Q}_\mu(g) + L(g). \qquad \square$$

We next turn to the compactness arguments.

Lemma 2.4

Suppose $\operatorname{supp}(\mu) \subset \mathbb{D}$. *Then there exists a compact operator $K_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ such that the equation*

$$g = \mathcal{P}_0(g) + K_\mu(g)$$

holds for all those $g \in H^{1/2}(\partial\mathbb{D})$ that satisfy $\mathcal{P}_\mu(g) = g$.

Proof. Suppose $g \in H^{1/2}(\partial\mathbb{D})$ and $\mathcal{P}_\mu(g) = g$. Then using Lemma 2.2 we can extend g to $\mathbb{D}$ so that

$$\overline{\partial} g = \mu \overline{\partial g} \quad \text{and} \quad g \in H^1(\mathbb{D}). \tag{2.15}$$

In particular, since $\operatorname{supp}(\mu) \subset B(0, r)$ for some $0 < r < 1$,

$$\mathcal{P}_0 g\,(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \text{ and } \mathcal{Q}_0 g\,(z) = \sum_{n=1}^{\infty} \hat{g}(-n) z^{-n}$$

are analytic for $|z| < 1$ and $r < |z|$, respectively. It follows that the function

$$G(z) = \begin{cases} g(z) - \mathcal{P}_0 g\,(z), & |z| < 1 \\ \mathcal{Q}_0 g\,(z), & r < |z| \end{cases} \tag{2.16}$$

is a well defined function with $G \in H^1_{\mathrm{loc}}(\mathbb{C})$. Also,

$$G(z) = \mathcal{C}(\overline{\partial} G)(z), \qquad z \in \mathbb{C}, \tag{2.17}$$

where

$$\mathcal{C}(h)\,(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(\omega)}{\omega - z}\, dm(\omega)$$

is the Cauchy transform.

Furthermore, since $\nabla G \in L^2(\mathbb{C})$we see that the derivatives ∂G and $\overline{\partial} G$ are connected through the equation

$$\partial G = \mathcal{S}\left(\overline{\partial} G\right) \tag{2.18}$$

where

$$\mathcal{S}g(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\omega)}{(\omega - z)^2}\, dm(\omega) \tag{2.19}$$

is the Beurling transform, a singular integral operator. It is well known and also easy to verify that $\mathcal{S}$ is an isometry on $L^2(\mathbb{C})$, see [1].

On the other hand, the above reasoning shows that $\overline{\partial} G = \chi_{\mathbb{D}}\, \overline{\partial} g$ in $\mathbb{C}$ while $\partial G = \partial g - (\mathcal{P}_0 g)'$ for $|z| < 1$. Thus combining (2.15) and (2.18) gives

$$\overline{\partial} G = \mu \overline{\partial g} = \mu \overline{\mathcal{S}(\overline{\partial} G)} + \mu \overline{(\mathcal{P}_0 g)'}.$$

As $\|\mu\|_\infty < 1$ and $\mathcal{S}$ is an L^2-isometry, we may rearrange the terms and obtain

$$\overline{\partial} G = \left(I - \mu\overline{\mathcal{S}}\right)^{-1}\left(\mu\overline{(\mathcal{P}_0 g)'}\right).$$

Inserting this expression to (2.17) shows that

$$g = \mathcal{P}_0 g \; + \mathcal{C}\Big[\left(I - \mu\overline{\mathcal{S}}\right)^{-1}\left(\mu\overline{(\mathcal{P}_0 g)'}\right)\Big], \qquad \text{on } \partial\mathbb{D}.$$

But

$$K_\mu h := \mathcal{C}\Big[\left(I - \mu\overline{\mathcal{S}}\right)^{-1}\left(\mu\overline{(\mathcal{P}_0 h)'}\right)\Big]$$

is compact as an operator

$$K_\mu : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$$

since

$$\left(I - \mu\overline{\mathcal{S}}\right)^{-1}\left(\mu\overline{(\mathcal{P}_0 h)'}\right) \in L^2(B(0,r))$$

whenever $h \in H^{1/2}(\partial\mathbb{D})$, and since $r < 1$. The lemma is proved. □

Corollary 2.5

Both of the operators

$$\mathcal{P}_\mu - \mathcal{P}_0 \circ \mathcal{P}_\mu \text{ and } \mathcal{Q}_\mu - \mathcal{Q}_0 \circ \mathcal{Q}_\mu$$

are compact on $H^{1/2}(\partial\mathbb{D})$.

Proof. The compactness of the first operator is clear since $\mathcal{P}_\mu = \mathcal{P}_0 \circ \mathcal{P}_\mu + K_\mu \circ \mathcal{P}_\mu$ by Lemma 2.4. For the other we use Lemma 2.3 and obtain for all $g \in H^{1/2}(\partial\mathbb{D})$

$$\begin{aligned}\mathcal{Q}_\mu(g) &= \overline{\mathcal{P}_\mu(\overline{g})} - L(g) = \overline{\mathcal{P}_0 \circ \mathcal{P}_\mu(\overline{g})} + K_1(g) \\ &= \mathcal{Q}_0\left(\overline{\mathcal{P}_\mu(\overline{g})}\right) + K_2(g) = \mathcal{Q}_0 \circ \mathcal{Q}_\mu(g) + K_3(g)\end{aligned} \tag{2.20}$$

where each $K_j : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$, $j = 1, 2, 3$, is a compact operator by Lemma 2.4. □

With these auxiliary results we can prove

Theorem 2.6

If $\operatorname{supp}(\mu) \subset \mathbb{D}$*, then*

$$\mathcal{P}_\mu - \mathcal{P}_0$$

is a compact operator on $H^{1/2}(\partial\mathbb{D})$.

Proof. The previous corollary shows that all $g \in H^{1/2}(\partial\mathbb{D})$ can be decomposed as

$$g = \mathcal{P}_\mu(g) + \mathcal{Q}_\mu(g) = \mathcal{P}_0 \circ \mathcal{P}_\mu(g) + \mathcal{Q}_0 \circ \mathcal{Q}_\mu(g) + K(g),$$

where K is compact. As $\mathcal{P}_0\mathcal{Q}_0 = 0$ we obtain

$$\mathcal{P}_0(g) = \mathcal{P}_0 \circ \mathcal{P}_\mu(g) + K_1(g)$$

where $K_1 : H^{1/2}(\partial\mathbb{D}) \to H^{1/2}(\partial\mathbb{D})$ is compact. Together with Corollary 2.5 this proves the claim. □

3. Fredholm properties

This subsection is devoted to the proof of Theorem 1.1. Recall first from [2] that for each compactly supported μ and each $k \in \mathbb{C}$ there exists a unique function $f = f_\mu(z) = f_\mu(z,k) \in H^1_{loc}(\mathbb{C})$ such that

$$f_\mu(z,k) = e^{ikz}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \text{ as } |z| \to \infty, \tag{3.1}$$

and that $f = f_\mu$ satisfies the differential equation $\overline{\partial} f = \mu \overline{\partial f}$. In fact, f_μ is locally Hölder continuous with gradient in $L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$.

The identity (3.1) tells that

$$\mathcal{P}_0(e^{-ikz} f_\mu) = 1$$

while Lemma 2.2 shows that $\mathcal{P}_\mu(f_\mu) = f_\mu$. Therefore by adding up we arrive at

$$f_\mu + e^{ikz} = \left(\mathcal{P}_\mu + \mathcal{P}_0^k\right) f_\mu,$$

proving part b) of Theorem 1.1. The subtraction would have led to another Fredholm equation, but it is not clear if the solution to this equation is unique.

Thus for the theorem, we are left to show that the operator $I - \left(\mathcal{P}_\mu^k + \mathcal{P}_0\right)$ is invertible. We start with

Theorem 3.1

The operator $I - (\mathcal{P}_\mu + \mathcal{P}_0^k)$ is injective on $H^{1/2}(\partial\mathbb{D})$.

Proof. Suppose $(\mathcal{P}_\mu + \mathcal{P}_0^k)f = f$, where $f \in H^{1/2}(\partial\mathbb{D})$. Operating by $\mathcal{P}_\mu$ on this identity we see that

$$\mathcal{P}_\mu \mathcal{P}_0^k(f) = 0.$$

We hence consider

$$g(z) = e^{ikz}\,\mathcal{P}_0(e^{-ikz}f)(z) = (\mathcal{P}_0^k\, f)(z).$$

This function extends analytically to $\mathbb{D}$. It also vanishes at origin since by (2.13)

$$L(g) = L(\mathcal{P}_\mu g) = 0. \tag{3.2}$$

This means that

$$G(z) := \overline{g(1/\overline{z})}$$

is analytic in the domain $\mathbb{C} \setminus \mathbb{D}$, with $G(z) \to 0$ as $z \to \infty$.

On the other hand, with Lemma 2.3 and (3.2) we have on $\partial\mathbb{D}$

$$\mathcal{Q}_\mu(\overline{g}) = \overline{\mathcal{P}_\mu(g)} = 0,$$

that is, $\mathcal{P}_\mu(\overline{g}) = \overline{g}$. Therefore Lemma 2.2 extends $G_{|\partial\mathbb{D}} = \overline{g}$ to $\mathbb{D}$ as an H^1-function satisfying

$$\overline{\partial} G = \mu \overline{\partial G}. \tag{3.3}$$

Since G is analytic outside $\mathbb{D}$, it satisfies this equation globally. It follows that $\nabla G \in L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$, see [1]. Hence G is locally Hölder continuous, vanishing at ∞. In particular, it is a bounded solution to (3.3) in the whole plane $\mathbb{C}$, therefore a constant by Liouville's theorem [1]. As G vanishes at ∞, $G \equiv 0$ and hence also $\mathcal{P}_0^k(f) = 0$ on $\partial\mathbb{D}$.

This last identity implies that f extends as an analytic function to $\mathbb{C} \setminus \mathbb{D}$ with

$$e^{-ikz} f(z) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{3.4}$$

Furthermore, it follows that $\mathcal{P}_\mu(f) = f$. Thus the function f extends to a H^1_{loc}-solution of $\overline{\partial} f = \mu \overline{\partial f}$ in $\mathbb{C}$. However, in [2, Theorem 4.2], we proved that a solution to (1.5) - (1.6) with development (3.4) must be identically zero. Thus $I - (\mathcal{P}_\mu + \mathcal{P}_0^k)$ is injective. □

To complete the proof of Theorem 1.1 we need to show

Theorem 3.2

On the space $H^{1/2}(\partial\mathbb{D})$ the operator $I - (\mathcal{P}_\mu + \mathcal{P}_0^k)$ is a Fredholm operator with index zero.

Proof. We write the operator in the form

$$I - (\mathcal{P}_\mu + \mathcal{P}_0^k) = \left(I - 2\mathcal{P}_0^k\right) + \left(\mathcal{P}_0^k - \mathcal{P}_\mu\right).$$

Therefore it suffices to show that

$$\left(I - 2\mathcal{P}_0^k\right) \text{ is a Fredholm operator with index zero} \tag{3.5}$$

and that

$$\left(\mathcal{P}_0^k - \mathcal{P}_\mu\right) \text{ is a compact operator on } H^{1/2}(\partial\mathbb{D})\,. \tag{3.6}$$

Now from (1.9),

$$\left(I - 2\mathcal{P}_0^k\right) h = -e^{ikz}\left(i\mathcal{H}_0 + L\right)(e^{-ik\cdot}h).$$

Since $\left(i\mathcal{H}_0 + L\right)^2 = I$, the claim (3.5) follows.

To prove the remaining claim write

$$\mathcal{P}_0^k - \mathcal{P}_\mu = \left(\mathcal{P}_0^k - \mathcal{P}_0\right) + \left(\mathcal{P}_0 - \mathcal{P}_\mu\right).$$

Here $\mathcal{P}_0 - \mathcal{P}_\mu$ is a compact operator by Theorem 2.6. For the compactness of the second factor

$$\mathcal{P}_0^k - \mathcal{P}_0 = e^{ikz}\mathcal{P}_0 e^{-ikz}\mathcal{Q}_0$$

note that $T(g) = \mathcal{P}_0\left(e^{-ikz}\mathcal{Q}_0(g)\right)$ is a Hankel operator with symbol e^{-ikz}, and such an operator with a continuous symbol is compact. Hence we have shown (3.6) and the theorem is proved. □

4. The non-linear Fourier transform

As a last theme we return to the Dirichlet to Neumann operators. It is a very useful observation that the boundary data corresponding to σ and $1/\sigma$ are equivalent.

Proposition 4.1

The Dirichlet to Neumann map Λ_σ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$ and $\Lambda_{1/\sigma}$.

Proof. We have shown in (2.8) for real valued u that

$$\partial_T \mathcal{H}_\mu(u) = \Lambda_\sigma(u) \tag{4.1}$$

holds in the weak sense. This is enough since by (2.10) $\mathcal{H}_\mu$ uniquely determines $\mathcal{H}_{-\mu}$. In fact, we have already used the identity

$$\mathcal{H}_{-\mu} - L = -(\mathcal{H}_\mu + L)^{-1}.$$

Note also that

$$\Lambda_{1/\sigma}(u) = \partial_T \mathcal{H}_{-\mu}(u) = -\partial_T (\mathcal{H}_\mu + L)^{-1}$$

since $-\mu = (1-\sigma^{-1})/(1+\sigma^{-1})$. □

In the previous sections we considered the solutions to (1.5) that have the exponential asymptotics (1.6). However, these give also the exponential solutions to the original divergence equation (1.1), through the representation

$$u(z) = \operatorname{Re} f_\mu(z) + i \operatorname{Im} f_{-\mu} = \frac{1}{2}\left(f_\mu + f_{-\mu} + \overline{f_\mu} - \overline{f_{-\mu}}\right).$$

Hence Proposition 4.1 implies that Theorem 1.1 gives an explicit algorithm also for finding the values of $u = u_k$ on $\partial\mathbb{D}$ and in the exterior disk.

Finally $\tau_\mu(k)$, the non-linear Fourier transform or the scattering coefficient of μ, can be obtained as follows; see [2] for details.The functions $m_\mu = e^{-ikz} f_\mu$ and $m_{-\mu} = e^{-ikz} f_{-\mu}$ are analytic outside the disk, with developments

$$m_\mu(z) = 1 + \frac{a_1^+(k)}{z} + \frac{a_2^+(k)}{z^2} + \cdots \quad \text{and} \quad m_{-\mu}(z) = 1 + \frac{a_1^-(k)}{z} + \frac{a_2^-(k)}{z^2} + \cdots$$

In [2, Section 5], it is shown that $\tau_\mu(k)$ satisfies the identity

$$\tau_\mu(k) = \frac{1}{2}\left(\overline{a_1^+(k)} - \overline{a_1^-(k)}\right), \qquad k \in \mathbb{C}.$$

Thus the above gives a constructive algorithm for finding this quantity.

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